Math 3235 Probability Theory

2/16/23

$$\sum_{i=1}^{c} (-1)^{i+1} \binom{c}{i} \frac{c}{i} = \quad \text{Pr. 7}$$

$$\sum_{i=0}^{c-1} \frac{c}{c-i}$$

Sums of Ind. r.v.

X_i that ind have the same dist

X_i are i.i.d. indepen. identically distributed

$E(X_i) = \mu \qquad Var(X_i) = \sigma^2$

N is another r.v. (integer)

$$S_N = \sum_{i=1}^{N} X_i$$

$$G_S(s) = G_N(G_{X_1}(s))$$

if X_i i.i.d. $\quad G_{X_i}(s) = G_{X_1}(s) = G_X(s)$

$$G_X(s) = \mathbb{E}(s^X)$$

$$G_X(1) = 1$$

$$G_X'(s) = \mathbb{E}\left(\frac{d}{ds}(s^X)\right) = \mathbb{E}\left(X s^{X-1}\right).$$

$$G_X'(1) = \mathbb{E}(X)$$

$$G_X''(1) = \mathbb{E}(X(X-1))$$

$$\frac{d^k}{ds^k} G_X(1) = \mathbb{E}(X(X-1)\cdots(X-k+1))$$

$$m_k(X) = \mathbb{E}(X^k)$$

k-Th moment of X

$$m_2(X) = G_X''(1) + G_X'(1)$$

$$\mathbb{E}(X(X-1)) + \mathbb{E}(X) =$$

$$= \mathbb{E}(X^2)$$

$$\mathrm{var}(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

X is geometric

$$\mathbb{P}(X = x) = (1-p)^{x-1} p$$

$$\mathbb{E}(s^X) = \sum_{x=1}^{\infty} s^x (1-p)^{x-1} p =$$

$$= \sum_{x=1}^{\infty} [s(1-p)]^{x-1} s p =$$

$$= \frac{sp}{1-s(1-p)} = G_X(s)$$

$$G_X(1) = 1$$

$$G_X'(s) = \frac{p}{1-s(1-p)} + \frac{sp(1-p)}{(1-s(1-p))^2}$$

$$G_X'(1) = \frac{1}{p} = \mathbb{E}(X)$$

$$var(X) = \frac{q}{p^2}$$

Again back to the random sums

$$S = \sum_{i=1}^{N} X_i \qquad X_i \text{ i.i.d}$$

$$G_S(s) = G_N(G_{X_1}(s))$$

$$G_S'(s) = G_N'(G_{X_1}(s))\, G_{X_1}'(s)$$

$$\mathbb{E}(S) = G_S'(1) = G_N'(G_{X_1}(1)) \cdot G_{X_1}'(1)) =$$

$$= G_N'(1)\, G_{X_1}'(1) =$$

$$\mathbb{E}(N)\,\mathbb{E}(X_1)$$

$X_0 \dots X_N$ are $+1$, -1
$\frac{1}{2}$ $\frac{1}{2}$

I stop when I see the first sequence of 3 +1.

N = the first time I see 3 H.

X_i are not independent from N.

if X_0, X_1, X_2 are all H or

$X_3\ X_4\ X_5$ are all H or

$\vdots$

$X_{3k}\ X_{3k+1}\ X_{3k+2}$ are all H.

$$S = \sum_{i=1}^{N} X_i$$

where $N = 3K+2$ is the last of the first 3 H appearing as

$X_{3k}, X_{3k+1}, X_{3k+2}$

$\mathbb{E}(S) = 0$!

$$Y_k = X_{3k} + X_{3k+1} + X_{3k+2}$$

Y_k $\qquad \mathbb{P}(Y_k = 3) = \frac{1}{8}$

$$\mathbb{E}(K) = 8$$

$$\mathbb{E}(Y_i \mid Y_i \neq 3)\, \overbrace{\mathbb{P}(Y \neq 3)}^{7/8} +$$

$$\mathbb{E}(Y_i \mid Y_i = 3)\, \underbrace{\mathbb{P}(Y_i = 3)}_{\frac{1}{8}} =$$

$$= \mathbb{E}(Y_i) = 0$$

$$\mathbb{E}(Y_i \mid Y_i = 3) = 3$$

$$\mathbb{E}(Y_i \mid Y_i \neq 3) = -\frac{3}{7}$$

$$\mathbb{E}(S) = 7 \cdot \left(-\frac{3}{7}\right) + 3 = 0 .$$

0

Chapter 5 Distribution and density function.

Arrival Time of a bus at a station.

t is a real number.

T random variable.

$\mathbb{P}(\text{Arrival time} = t) = 0$

Let's Take a very short Time interval dt

$$\mathbb{P}(t \le \text{Arrival} \le t + dt) = f(t)\,dt$$

$f(t)$ is called the prob. density function

$$\mathbb{P}(t_1 \leq \text{Arrival} \leq t_2) = \int_{t_1}^{t_2} f(t)\, dt$$

X is a continuous r.v. with p.d.f. f_X if

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\, dx$$

$$\parallel$$

$$\mathbb{P}(a < X \leq b)$$

$X : \Omega \rightarrow \mathbb{R}$ random variable

(cumulative) dist. function F_X

$\rightarrow$ $F_X(x) = \mathbb{P}(X \leq x)$

$\rightarrow$ $A_x = \{ \omega \mid X(\omega) \leq x \} \in \mathcal{F}$

If X is a r.v. Then c.d.f. is defined.

$$F_X(x) \geq 0$$

$$\{X \leq x\} \subset \{X \leq y\}$$

$$y \geq x$$

$$F(y) \geq F(x) \qquad y \geq x$$

F is non decreasing

$$F(-\infty) = \mathbb{P}(X \leq -\infty) = 0$$

$$\lim_{x \to -\infty} F(x) = 0$$

$$\bigcap_{n=0}^{\infty} \mathbb{P}(X \leq -n) = \phi$$

$$\lim_{x \to \infty} F(x) = 1$$

if F_X is the c.d.f. of a r.v. X then

1) $F_X(x) \geq 0$

2) $F_X(y) \geq F_X(x)$ if $y \geq x$

3) $\lim_{x \to -\infty} F_X(x) = 0$ $\quad \lim_{x \to +\infty} F(x) = 1$

If X is a discrete r.v., how does F_X look like?

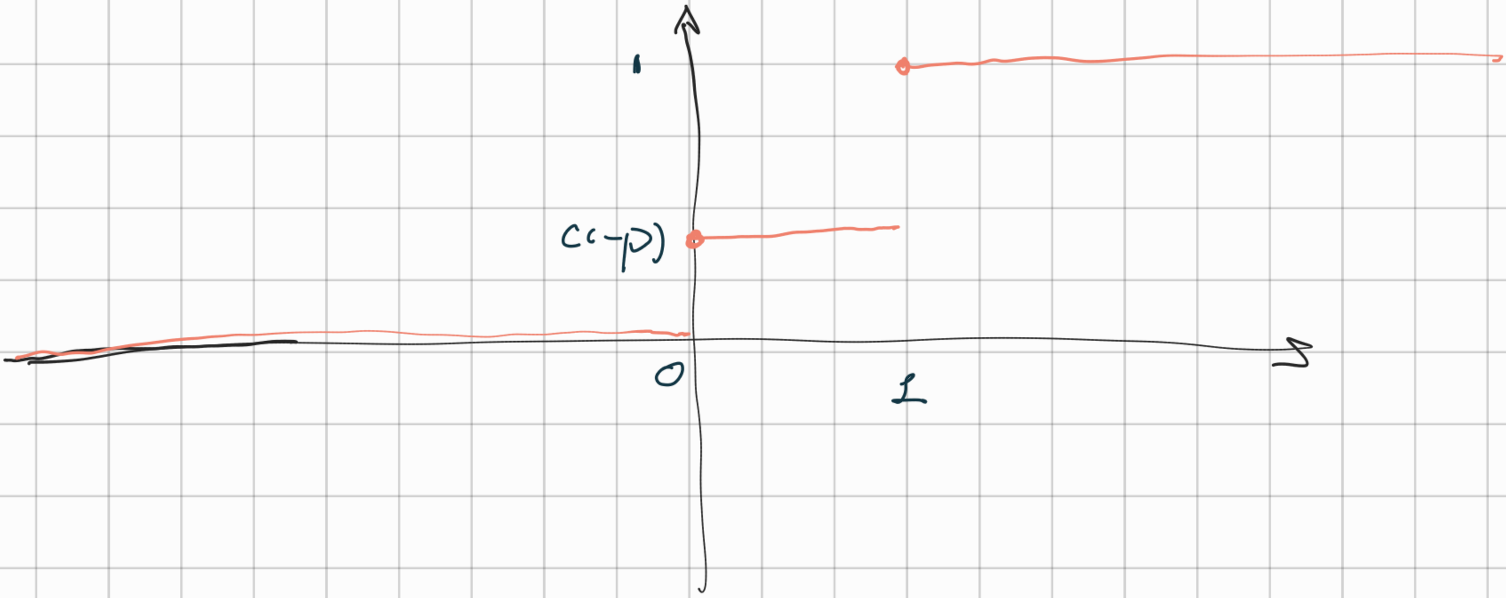
X Bernoolli $\quad$ $0 \quad q = (1-p)$

$\quad\quad\quad\quad\quad\quad 1 \quad p$

if $x < 0$ $\quad F_X(x) = 0$

$0 \leq x < 1$ $\quad F_X(x) = (1-p)$

$x \geq 1$ $\quad F_X(x) = 1$



$$F_X(0) = \mathbb{P}(X \leq 0) = (1-p)$$

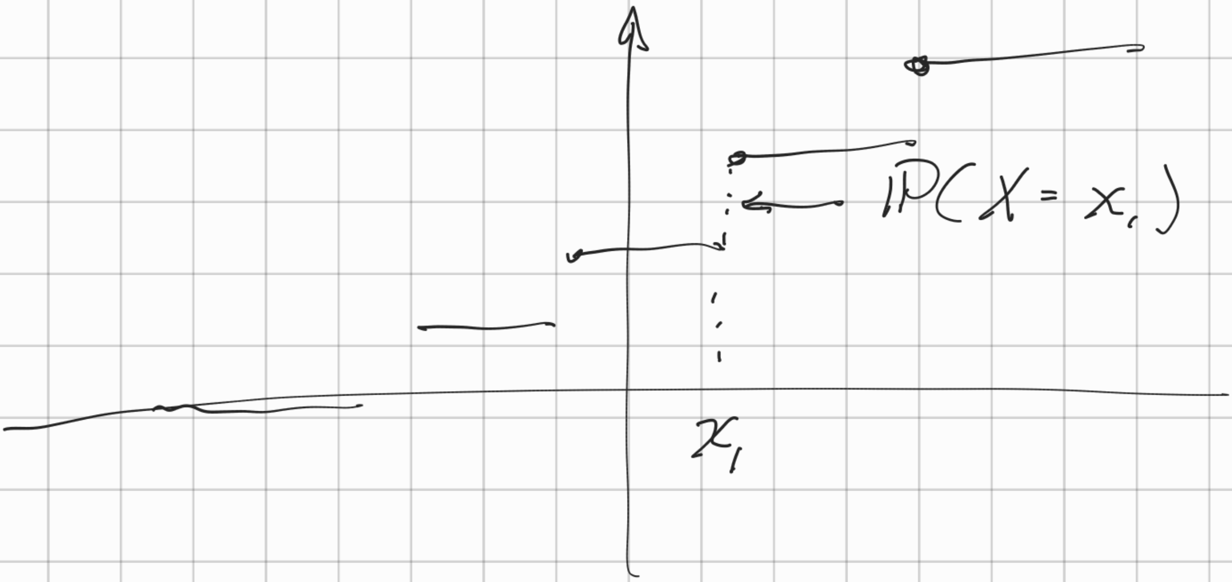
4) F_X is continuous from the right.

Property: X is a discrete r.v. if F_X is piecewise constant.

That is F_X has at most countably many discontinuity x_k that form the $\mathrm{Im}(X)$.

$$\mathbb{P}(X = x_k) = F(x_k) - F(x_k^-)$$

$$F(x_k^-) = \lim_{x \to x_k^-} F(x)$$



$$\mathbb{P}(X = x_k) = \mathbb{P}(X \leq x_k) - \mathbb{P}(X \leq x_{k-1})$$
$$= F(x_k) - F(x_{k-1})$$

if x_k are ordered.